FUNCTORIALITY

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First, the class of groups we work with are connected reductive algebraic groups, for example $G = \mathrm{GL}_n$. In general, we would work over a global field but today we work over k a number field, and let $\mathbb{A}_k$ be the adeles over k.

Second, we consider automorphic forms: from f, a complex-valued function on the upper half-plane, we define a complex-valued function ϕ_f on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. Letting $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ act, we end up with a representation, so this sits naturally in the space

$$L^2(ZG(k)\backslash G(\mathbb{A}_k), \omega)$$

of square-integrable functions which transform under the center Z according to the character ω. We look inside the space L^2_0 of cusp forms, vanishing at the cusps: let $P = MN \subseteq G$ be a Levi subgroup, and insist that

$$\int_{N(k)\backslash N(\mathbb{A})} \phi(ng)\, dn = 0$$

for all $g \in G$. A representation π in L^2_0 decomposes as a tensor product $\pi = \otimes_v \pi_v$ where π_v is a representation of $G(k_v)$, so we need to understand representations of groups over local fields; this is necessary, but not sufficient, as automorphic forms contain some global constraints and not all local representations will necessarily occur. These are often deep questions.

Third, we consider L-groups. Langlands computed constant terms of Eisenstein series and saw a natural factorization, and this led him to the notion of L-group. Let T be a maximal torus over $\overline{k}$. Let $X = X^*(T)$ be the group of characters of $T(\overline{k})$, and let $X^\vee = X_*(T)$ be the group of cocharacters, maps from $\overline{k}^* \to T(\overline{k})$. Let Σ be the roots of T, the nonzero eigenfunctions of $T(\overline{k})$ on the Lie algebra $\mathfrak{g}(\overline{k})$. Dually, there is also a notion of coroots $\Sigma^\vee$.

From this we have the data $(X, \Sigma, X^\vee, \Sigma^\vee)$. Let $\Delta \subseteq \Sigma$ the simple roots; then we also have the data $(X, \Delta, X^\vee, \Delta^\vee)$, from which one can recover the group. Taking the dual data $(X^\vee, \Sigma^\vee, X, \Sigma)$, we obtain a corresponding dual group $G^\vee$ the dual group, a complex group. For example, we can work things out explicitly for $G = \mathrm{SL}_2$, and we find that the dual group gives us back SL_2.

Suppose now that G is semisimple, so $G = G_{\mathrm{der}}$, finite center. Let $C = (\langle \alpha_i, \alpha_j \rangle)$ the Cartan matrix, where

$$\langle \alpha_i, \alpha_j \rangle = \frac{2\kappa(\alpha_i, \alpha_j)}{\kappa(\alpha_j, \alpha_j)}.$$

The transpose ${}^tC = C^\vee$ the matrix for the dual data. For $G = \mathrm{Sp}_{2n}$ we have $G^\vee = \mathrm{SO}_{2n+1}$ and vice versa. Over a nonalgebraically closed field, we keep track of also the Galois action, and we define ${}^LG = G^\vee \rtimes \Gamma_k$ where $\Gamma_k = \mathrm{Gal}(\overline{k}/k)$, or sometimes replacing Γ_k by the Weil group.

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Let R be the root lattice, the $\mathbb{Z}$-lattice generated by the roots in $\mathbb{R}^n$ (n is then the rank of the semisimple group G). Let Q be the weight lattice, $\chi \in \mathbb{R}^n$ such that $\langle \chi, \alpha \rangle \in \mathbb{Z}$ for all $\alpha \in R$. We have $Q \supseteq X \supseteq R$. If G is simply connected, then $X = Q$; if G is adjoint, then $X = R$.

We then have the following table.

G	$G^\vee$
GL_n	$\mathrm{GL}_n(\mathbb{C})$
SO_{2n}	$\mathrm{SO}_{2n}(\mathbb{C})$
SO_{2n+1}	$\mathrm{Sp}_{2n}(\mathbb{C})$
GSpin_{2n}	$\mathrm{GSO}_{2n}(\mathbb{C})$
GSpin_{2n+1}	$\mathrm{GSp}_{2n}(\mathbb{C})$
$\mathrm{GSpin}_5 = \mathrm{GSp}_4$	$\mathrm{GSp}_4(\mathbb{C})$

The latter is because of an accidental isomorphism.

Now we discuss unramified representations. Let K be a maximal compact subgroup of $G(k_v)$, where $k = k_v$ is a p-adic field. Let T be a maximal torus. By the Iwasawa decomposition, we have $G(k) = T(k)K$. Let π be an irreducible admissible representation of $G(k)$; we say that π is *unramified* (or *spherical*) if there exists a w in the representation space of π that is fixed by K; in patricular, if π is irreducible then there is a unique w that is 1-dimensional. If π is unramified, then π is a constituent of $I(\chi)$ (induced representation), χ a character of $T(k)$, so

$$I(\chi) = \{f(tug) = \chi(t)\delta^{1/2}(t)f(g) : f \text{ is smooth on } G\}.$$

If χ is unramified, then χ restricted to ${}^oT(k) = T(k) \cap K$ is identically 1; so unramified characters are homomorphisms

$$\Lambda = T(k)/{}^oT(k) \to \mathbb{C}^*.$$

Let X_{un} be the group of characters of Λ, and let $\Lambda = X^\vee(T) = X(T^\vee)$. Then

$$\mathrm{Hom}(\Lambda, \mathbb{C}^*) = \mathrm{Hom}(X(T^\vee), \mathbb{C}^*) = T^\vee.$$

So $T^\vee$ then becomes the main object of study in defining L-functions.

The conclusion: unramified characters are parametrized by $T^\vee$; let W be the Weyl group, acting $\chi^w(t) = \chi(w^{-1}tw)$, then

$$X_{\mathrm{un}}/W \leftrightarrow T^\vee/W^\vee = G^\vee\text{-conjugacy classes of } T^\vee.$$

Thus $I(\chi)$ and $I(\chi^w)$ have the same constituents.

Langlands realized, by interpreting constant terms of Eisenstein series, that if r is a finite-dimensional representation of ${}^LG$, and $s \in \mathbb{C}$, since from π_v we obtain a conjugacy class $C(\pi_v)$ in $T^\vee/W^\vee$, and so we can write down the unramified L-function

$$L(s, \pi_v, r) = \det(1 - r(C(\pi_v))q_v^{-s})^{-1}$$

where q_v is the number of elements in the residue field of k.(Unfortunately, there are a lot more cuspidal representations π of $G(\mathbb{A}_k)$ than there are ρ representations of W_k.)

Let $f(z) = \sum_{n=1}^\infty a_n e^{2\pi i n z}$ be a cusp form on the classical group $\Gamma_0(N)$ of weight k. Let

$$a_n = p^{(k-1)/2}(\alpha_p + \alpha_p^{-1}).$$

Then f corresponds to a representation $\pi = \otimes_p \pi_p$, and π_p corresponds to the conjugacy class $\begin{pmatrix} \alpha_p & 0 \\ 0 & \alpha_p^{-1} \end{pmatrix}$; if r is the standard representation of $\mathrm{GL}_2(\mathbb{C})$, then the unramified L-function is

$$(1 - \alpha_p p^{-s})^{-1}(1 - \alpha_p^{-1} p^{-s})^{-1}.$$

Now we can define functoriality. Let G, G' be connected reductive groups, where G' is quasisplit. Suppose we have a homomorphism $\phi : {}^L G \to {}^L G'$. Let $\pi = \otimes_v \pi_v$ be a cuspidal representation of $G(\mathbb{A}_k)$; recall that at almost all primes, v, the representation π_v corresponds to conjugacy classes $C(\pi_v)$. Then we can consider the image $\phi(\{C(\pi_v)\})$ of conjugacy classes in ${}^L G'$; we can ask does there exist a cuspidal automorphic representation $\Pi = \otimes_v \Pi_v$ an automorphic representation on $G'(\mathbb{A}_k)$ that agrees with these conjugacy classes at almost all v? Whether or not this appears in an L^2-space is very deep question. Functoriality implies equality of L-functions, root numbers, etc. We want further to have the second projetion of $\phi(x, w)$ is still w, so we require the map ϕ to be a so-called *L-homomorphism.*

Most of the time, $G' = \mathrm{GL}_n$; in this case, by strong multiplicity one, if such a representation Π exists, then it is necessarily unique. In other cases, this representation may not be unique, so we group them together into L-packets.

Here are some examples. Symmetric powers of GL_2. Then $G = \mathrm{GL}_2$, and consider

$$\phi = \mathrm{Sym}^m : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_{n+1}(\mathbb{C})$$

defined as follows: if $P(X,Y)$ is a form of degree m and $g \in \mathrm{GL}_2(\mathbb{C})$ then $P_1(X,Y) = P((X,Y)g)$, and expressing coefficients gives you $\mathrm{Sym}^m g$. In this context, the question of functoriality asks: is there a functorial transfer Sym^m of representations?

Theorem. Sym^m *is functorial for $m = 2$ (Gelbart–Jacquet, 1978), $m = 3$ (Kim–Shahidi, 2002), $m = 4$ (Kim, 2002).*

Unfortunately, at this point the image (of the local Galois representations) may not be solvable, so it is likely to be very hard; and the depth of functoriality would have many important consequences.

Now we also have the local Langlands correspondence by GL_n by Harris–Taylor, Henniart, Scholze; this allows you to make local candidates, so the hard part is to prove that the corresponding global candidate is indeed automorphic.

Over k_v, the representation π_v gives ρ_v a two-dimensional representation of W_{k_v}, so we have a homomorphism

$$W_{k_v} \xrightarrow{\rho_v} \xrightarrow{\mathrm{Sym}^m} \mathrm{GL}_{m+1}(\mathbb{C}).$$

and by the local Langlands correspondence, $\mathrm{Sym}^m \pi_v$ corresponds to $\mathrm{Sym}^m \rho_v$. Then the question is whether or not $\mathrm{Sym}^m \pi_v$ is automorphic; and it is known for $m = 2, 3, 4$.

Suppose π comes from a Galois group. Then π corresponds to $\rho : \Gamma \to \mathrm{GL}_2(\mathbb{C})$, and we can postcompose using Sym^m to land in $\mathrm{GL}_{m+1}(\mathbb{C})$. For the representation $\mathrm{Sym}^m \rho$ of Γ, this *does* come from an automorphic representation of $\mathrm{GL}_{m+1}(\mathbb{A})$ for all m.

For π_v corresponding to the semisimple conjugacy class $\begin{pmatrix} \alpha_v & 0 \\ 0 & \beta_v \end{pmatrix}$ of $\mathrm{GL}_2(\mathbb{C})$, then

$$\mathrm{Sym}^m \begin{pmatrix} \alpha_v & 0 \\ 0 & \beta_v \end{pmatrix} = \mathrm{diag}(\alpha_v^m, \alpha_v^{m-1}\beta_v, \ldots, \beta_v^m).$$

By work of Luo–Rudnick–Sarnak, the Ramanujan conjecture asserts $|\alpha_v| = |\beta_v| = 1$, and they prove $q_v^{-5/34} < |\alpha_v|, |\beta_v| < q_v^{5/34}$.

If $\mathfrak{H}$ is the upper half-plane and Γ is a congruence subgroup, inside $L^2(\Gamma\backslash\mathfrak{H})$, for the Laplace operator $\Delta = -y^2(d^2/dx^2 + d^2/dy^2)$, Selberg conjectured that the smallest nonzero eigenvalue satisfies

$$1/4 \leq \lambda_1(\Gamma\backslash\mathfrak{H});$$

the best known result is due to Blomer–Brumely 2012 that

$$\lambda_1(\Gamma\backslash\mathfrak{H}) \geq 1/4 - (7/64)^2 = 0.238;$$

this already leads to many nice results. And we have

$$q_v^{-7/64} < |\alpha_v|, |\beta_v| < q_v^{7/64}.$$

This was all just for GL_2. Now let G be a group whose L-group is classical, so $G = \mathrm{SO}, \mathrm{Sp}, \mathrm{U}, \mathrm{GSpin}$. Then ${}^LG \hookrightarrow \mathrm{GL}_N(\mathbb{C})$, and functoriality is proven in these cases. Using the trace formula, these have been proven by Arthur for the first two $G = \mathrm{SO}, \mathrm{Sp}$, and by Mok and others for $G = \mathrm{U}$. This has consequence for the entirety of the corresponding L-functions: for example, if π is a repressentation of Sp_4, then the L-functions $L^S(s, \pi, \mathrm{spin})$ of degree 4 and $L^S(s, \pi, \mathrm{std})$ of degree 5 are entire, and this extends to several symmetric powers as well.

So the Langlands philosophy indicates that we should use the trace formula to further investigate functoriality.